ON THE POSSIBLE ROLE OF HORIZONTAL DIVERGENCE TO MEANDERS OF A WIDE CURRENT IN A STRATIFIED OCEAN

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Received November 20, 1959.

Abstract

Expansion of H. Stommel's theory for the meanders of a very wide current in a strtified ocean is studied. And it is shown that if the thickness of the lower layer is limited or the current exists there, the meanders may occur in the ocean of which upper current is smaller than Stommel's critical value.

1. Introduction

The Gulf Stream meanders observed on the multiple ship survey of June, 1950(Fuglister and Worshington, 1951) were given attention by oceanographers and inspired several theoretical studies. Uda (1949, 1951) investigated the same patterns of the Kuroshio. Haurwitz and Panofsky (1950) showed the existence of unstable waves with reasonable velocities of propagation considering several modes of currents in a homogeneous ocean with cross-stream velocity profiles similar to those observed in the true Gulf Stream and these waves are a result of the shearing instability. There are some Japanese oceanographers who applied this theory to the Kuroshio.

H. Stommel(1953) presented to us an interesting paper about the Gulf Stream meanders. He studied the meanders for a very wide current in a stratified two-layer ocean. According to his theory, the certain types of meanders might exist in which stratification and inertia are dynamically important. He assumed that the lower layer is very deep and thence that the horizontal pressure gradients vanish in it at all times. His results are that for upper layer-current velocity $U_1^2 < g'D_1$, all waves are stable and at $U_1^2 = g'D_1$, a single wave number given by $k = f/2U_1$ becomes "just unstable", whereas all other waves are stable and for slightly larger values of U_1^2, there is a narrow range of wave numbers about $k = f/2U_1$ in which waves are unstable.

However, if we assume the thickness of upper layer D_1 is 200m. and the density difference of two layers 2×10^{-3}, the meanders can not be expected unless $U_1 >$ 200cm/sec. But to the author this value seems to be slightly larger and the wave patterns can be seen not only in the homogeneous upper layer but also in the lower layer below thermocline. If we treat the model in which the ocean depth is limited or the currents exist in the lower layer, the unstable waves might exist even though $U_1^2 < g'D_1$.

2. A Meander Theory for a Very Wide Current in a Stratified Ocean

To simplify the analysis, we consider the almost same model of ocean as that of Stommel. That is, in the undisturbed state steady basic currents U_1 and U_2 flow in the x-direction in the upper and lower layer respectively and their thicknesses are D_1 and D_2 and the density difference of two layers $\Delta\rho=\rho'-\rho$. Then associated with these currents are cross stream pressure gradients of the following forms:

$$fU_1=-g\frac{\partial\zeta_1}{\partial y}$$

$$f\frac{\rho'}{\Delta\rho}\left(\frac{\rho}{\rho^1}U_1-U_2\right)=g\frac{\partial\zeta_2}{\partial y}$$

where f is Colioris parameter $2\,\omega\,\sin\varphi$ and ζ_1, ζ_2 are the elevations of sea surface and interface.

Now we suppose that there are small perturbations u_1, v_1, u_2, v_2, in the velocity components and ζ_1', ζ_2', in the elevations of the free surface and interface and that these quantities are independent of y, the cross-stream coordinate. The perturbation equations can be written in the form

For upper layer

$$\frac{\partial u_1}{\partial t}+U_1\frac{\partial u_1}{\partial x}-fv_1=-g\frac{\partial\zeta_1'}{\partial x} \tag{1}$$

$$\frac{\partial v_1}{\partial t}+U_1\frac{\partial v_1}{\partial x}+fu_1=0 \tag{2}$$

$$\frac{\partial}{\partial t}(\zeta_1'-\zeta_2')+U_1\frac{\partial}{\partial x}(\zeta_1'-\zeta_2')+v_1\left(\frac{\partial\zeta_1}{\partial y}-\frac{\partial\zeta_2}{\partial y}\right)+D_1\frac{\partial u_1}{\partial x}=0 \tag{3}$$

For lower layer

$$\frac{\partial u_2}{\partial t}+U_2\frac{\partial u_2}{\partial x}-fv_2=-g\frac{\rho}{\rho'}\frac{\partial\zeta_1'}{\partial x}-g\frac{\Delta\rho}{\rho'}\frac{\partial\zeta_2'}{\partial x} \tag{4}$$

$$\frac{\partial v_2}{\partial t}+U_2\frac{\partial v_2}{\partial x}+fu_2=0 \tag{5}$$

$$\frac{\partial\zeta_2'}{\partial t}+U_2\frac{\partial\zeta_2'}{\partial x}+v_2\frac{\partial\zeta_2}{\partial y}+D_2\frac{\partial u_2}{\partial x}=0 \tag{6}$$

If the perturbations are all in the form $e^{i(kx-\sigma t)}$ the following frequency equation is obtained from the above equations,

$$\begin{aligned}
&p^6-3(1+\delta)p^5-\{(1+\varepsilon)a^2+2l^2-3(1+3\delta+\delta^2\}p^4\\
&+[4(1+\delta)l^2+\{(3+\delta)\varepsilon+1+3\delta\}a^2-(1+\delta)(1+8\delta+\delta^2)p^3\\
&+\left\{(1+\varepsilon)a^2l^2+\varepsilon a^4\frac{\Delta\rho}{\rho'}-3(+\delta)^2l^2-3(1+\delta)(\varepsilon+\delta)a^2-3(1-\delta)^2l^2\frac{\rho'}{\Delta\rho}\right.\\
&\left.+3\delta(1+3\delta+\delta^2)\right\}p^2+\left[-(1+\delta)a^4\varepsilon\frac{\Delta\rho}{\rho'}-2l^2a^2(\delta\varepsilon+1)+3l^2\frac{\rho'}{\Delta\rho}(1-\delta)(1-\delta^2)\right.\\
&\left.+(1+\delta)^3l^2+\{(1+3\delta)\varepsilon+\delta^2(3+\delta)\}a^2-3\delta^2(1+\delta)\right]p\\
&+\left\{(1+\delta^2\varepsilon)l^2a^2+\delta a^4\varepsilon\frac{\Delta\rho}{\rho'}-(1-\delta)(1-\delta^3)l^2\frac{\rho'}{\Delta\rho}-\delta(1+\delta^2)l^2\right.\\
&\left.-\delta(\varepsilon+\delta^2)a^2+\delta^3\right\}=0
\end{aligned} \tag{7}$$

where $p=\frac{c}{U_1}=\frac{\partial}{kU_1}$, $\delta=\frac{U_2}{U_1}$, $a^2=\frac{gD_1}{U^2}$, $l^2=\frac{f^2}{k^2U_1^2}$, $\varepsilon=\frac{D_2}{D_1}$

To simplify the above frequency equation, we examine the order of magnitude of each term.

Then the order of $a^2 \sim 10^3$ and generally $|\delta|<1$, $\varepsilon>1$,

Consequently, the coefficients of p^6, p^5 are very small compared to the others. So the simplified frequency equation can be written in the form

$$-(\varepsilon+1)p^4+\{1+3\delta+(3+\delta\varepsilon\}p^3+\{-3(1+\delta)(\varepsilon+\delta)-3(1-\delta)^2\frac{l^2}{\alpha}+\alpha^2\varepsilon$$
$$+l^2(\varepsilon+1)\}p^2+[(1+3\delta)\varepsilon+3(1-\delta)\frac{l^2}{\alpha^2}-(1+\delta)\varepsilon\alpha^2-2(\varepsilon\delta+1)l^2]p$$
$$+\{-\delta\varepsilon-(1-\delta)\frac{l^2}{\alpha^2}+\delta\varepsilon\alpha^2+(\varepsilon\delta^2+1)l^2\}=0 \tag{8}$$

where $\alpha^2=a^2\frac{\Delta\rho}{\rho'}=\frac{gD_1}{U_1^2}\frac{\Delta\rho}{\rho'}$

In $p=p_1+ip_2$ of the above equation (8), the real part p_1 of which is the ratio of the velocity of propagation of the wave to the velocity of the upper current, and the imaginary part p_2 of which gives the instability of the wave motion. Since the equation (8) is the quadric equation of p, we can rewrite it to the form

$$x^4+px^2+qx+r=0 \tag{9}$$

The necessary and surfficient conditions of the above equation (9) having four real roots are

$$p<0, \quad p^2-4r>0, \quad D\geqq 0 \tag{10}$$

where $D=16r(p^2-4r)^2-4pq^2(p^2-36r)-27q^4$

However it is rather complicated works to study the conditions (10). So we will deal with several simple cases.

Case I. Non lower-current ($\delta=0$)

For the first time, in order to study the effect of ratio of thickness of upper layer to that of lower layer, we assume $\delta=0$ in the equation (8), then we get

$$-(\varepsilon+1)p^4+(3\varepsilon+1)p^3+\left\{\alpha^2\varepsilon+(\varepsilon+1)l^2-3\frac{l^2}{\alpha^2}-3\varepsilon\right\}p^2$$
$$+\left\{\varepsilon+3\frac{l^2}{\alpha^2}-\varepsilon\alpha^2-2l^2\right\}p+l^2\left(1-\frac{1}{\alpha^2}\right)=0 \tag{11}$$

If ε tends to ∞ in this equation, the equation (11) becomes that of Stommel in his paper. As a simple case, suppose to be $\alpha^2=\frac{gD_1}{U_1^2}\frac{\Delta\rho}{\rho'}=1$ which is Stommel's critical value, the equation (11) becomes the following cubic equation of p

$$-(\varepsilon+1)p^3+(3\varepsilon+1)p^2+\{l^2(\varepsilon-2)-2\varepsilon\}p+l^2=0 \tag{12}$$

In this case, our problem is reduced to the discriminant of roots of the equation (12). That is, if the discriminant of eq. (12) $D<0$, we can obtain a pair of conjugate imaginary roots and there exist unstable waves of which amplitude might be expected to grow so large as to be noticeable by hydrographic observations.

1) $\varepsilon=5$

In this case, the ocean is rather shallow and 6 times as deep as the thickness of the homogeneous surface layer.

The discriminant of the equation (12) is

$$D=l^6-15.944\,l^4+11.012\,l^2+2.469$$

So that, if $0.90<l^2<15.20$, $D<0$. Consequently suppose $U_1=200$ cm/sec., $\varphi=30°$N, then the disturbances with wave lengths 147 km—603 km are unstable.

2) $\varepsilon=10$

In this case, if $1.10<l^2<6.20$, $D<0$. That is, for the same values of U_1 and φ as the above, the waves lengths of unstable waves are 163 km—387 km.

3) $\varepsilon=20$

If 1 $2.5<l^2<4.05$, $D<0$. That is, the unstable wave lengths are 170 km—315 km.

From these results, we may conclude that the shallower the depth of ocean is, the larger the region of the unstable waves is. In other words, the meanders will easily occur.

3. Calculations of the Period or the Velocity of Propagation and Intensity of Unstable Waves ($\alpha^2=1$)

We can obtain the velocity of propagation of unstable waves by solving the frequency equation. The results are shown in Table 1.

TABLE 1. THE PERIODS AND INSTABILITIES OF UNSTABLE WAVES.

l^2 \ ε	5		10		20	
	p_1	p_2	p_1	p_2	p_1	p_2
1	0.252	0.117	—	—	—	—
2			0.047	0.244	0.027	0.181
3	−0.004	0.432	−0.079	0.283	−0.114	0.180
4			−0.189	0.287	—	—
5	−0.188	0.491	−0.290	0.206	—	—
6			−0.380	0.029	—	—
10	−0.513	0.456	—	—	—	—
15	−0.766	0.092	—	—	—	—

As is seen from Table 1, for a certain value of ε, we can except a wave that does not propagate but its amplitude grows with time.

Now, we assume that the thickness of upper layer $D_1 = 200\text{m}$ and $\frac{\Delta\rho}{\rho'} = 2\times10^{-3}$, then $U_1 = 200$ cm/sec for $\alpha^2=1$.
So, for $\varepsilon=20$, we obtain the velocity of propagation of unstable waves

$$c = 5.4 \text{ cm/sec} \qquad \text{for } l^2=2$$
$$c = -22.8 \text{ cm/sec} \qquad \text{for } l^2=3$$

and as to the period of unstable waves

$$T = 52.4 \text{ days} \qquad \text{for } l^2=2$$
$$T = 15.2 \text{ days} \qquad \text{for } l^2=3$$

And these values may be reasonable. And the unstable wave with wave length $l^2=2$ propagates toward the same direction at that of the upper layer currents and the unstable wave with $l^2=3$ propagates toward the opposite direction of upper layer currents.

We will next obtain the time in which the amplitude will be e^2 (=7.39) times as large as that of initial disturbance,

$$T = 2.16\times10^5 \text{ sec} = 2.5 \text{ days} \qquad \text{for } l^2=2$$
$$T = 2.64\times10^5 \text{ sec} = 3.1 \text{ days} \qquad \text{for } l^2=3$$

From this calculations, we can see that the unstable waves will grow to a few times as larger as that of initial disturbance in a few days.

4. Non Propagating Unstable Waves

Next, we consider non propagating unstable waves. First we consider $\delta=0$. In the case of non propagating unstable waves, the equation (8) has pure imaginary roots $p=\pm i p_0$. So, substitute from this into the equation (8), then we get

$$\begin{cases} -(\varepsilon+1)p_0^4 + \left\{\varepsilon(\alpha^2+l^2-3) + l^2\left(1-\frac{3}{\alpha^2}\right)\right\}p_0^2 - l^2\left(1-\frac{1}{\alpha^2}\right) = 0 \\ (3\varepsilon+1)p_0^3 - \left\{\varepsilon(1-\alpha^2) + l^2\left(\frac{3}{\alpha^2}-2\right)\right\}p_0 = 0 \end{cases}$$

From the second equation, the following equation is obtained

$$p_0^2 = \frac{\varepsilon(l-\alpha^2) + l^2\left(\frac{3}{\alpha^2}-2\right)}{(3\varepsilon+1)}$$

Moreover $p_0^2>0$ should be satisfied. If $\alpha^2<1$, the condition of $p_0^2>0$ is always satified and if $1<\alpha^2<1.5$ and $l^2>\frac{\varepsilon\alpha^2(\alpha^2-1)}{3-2\alpha^2}$ the unstable waves exist.

In the special case $\alpha^2=1$, and unstable wave with wave length

$$l^2 = \frac{2\varepsilon}{\varepsilon-2+\frac{\varepsilon+1}{3\varepsilon+1}}$$

exist. Namely,

$$l^2=2.96 \qquad \text{for } \varepsilon=5$$
$$l^2=2.39 \qquad \text{for } \varepsilon=10$$
$$l^2=2.19 \qquad \text{for } \varepsilon=20$$

and when ε tends to ∞, l^2 becomes 2 which Stommel got in his paper. The results for several values of α^2 are shown in Table 2.

TABLE 2. WAVE LENGTHS AND INSTABILITIES OF NON PROPAGATING UNSTABLE WAVES.

ε / α^2	5		10		20	
	l^2	p_0	l^2	p_0	l^2	p_0
1	2.96	0.429	2.39	0.277	2.19	0.196
1.2	4.30	0.268	6.10	0.184	10.8	0.151
1.4	23.0	0.284	36.3	0.195	—	—

From the Table 2, we can easily see that the shallower the ocean is, the greater instability the unstable waves have.

Next we consider the effect of lower layer currents. That is, we put $\varepsilon\to\infty$, $\delta\neq 0$ in the equation (11) then the following equation is given

$$-p^4+(3+\delta)p^3+\{-3(1+\delta)+\alpha^2+l^2\}p^2+[(1+3\delta)-(1+\delta)\alpha^2-2\,\delta l^2]p$$
$$+(-\delta+\delta\alpha^2+\delta^2 l^2)=0$$

Similarly, we put $p=\pm i p_0$ in the above frequency equation, then we get

$$p_0^4-\{3(1+\delta)-\alpha^2-l^2\}p_0^2+\{\delta(1-\alpha^2)-\delta^2 l^2\}=0$$
$$p_0^2=\frac{(1+3\,\delta)-(1+\delta)\,\alpha^2-2\,\delta l^2}{(3+\delta)}>0$$

The Table 3 shows the wave lengths and instabilities of unstable waves for various values of δ and α^2.

TABLE 3. WAVE LENGTHS AND INSTABILITIES OF UNSTABLE WAVES FOR VARIOUS VALUES OF δ AND α^2.

α^2 / δ	1		1.1		1.2		1.3		1.4	
	l^2	p_0	l^2	p_0	l^2	p_0	l^2	p_0	l^2	p_0
0	2	0	—	—	—	—	—	—	—	—
−0.1	1.93	0.255	1.87	0.170	—	—	—	—	—	—
−0.2	1.88	0.354	1.81	0.295	1.76	0.227	1.68	0.014	—	—
−0.3	1.82	0.427	1.76	0.387	1.70	0.312	1.63	0.249	1.58	0.151

From this Table 3, if the current of lower layer flows toward the opposite direction to that of upper layer the unstable waves can be expected even though the current of upper layer is slower than $U_1^2=gD_1\frac{\Delta\rho}{\rho'}$.

5. Conclusion

In this paper, it has been shown that the unstable waves can be expected even though the upper layer current velocity is smaller than Sommel's critical value if the ocean has finite depth and the currents in the lower layer flow in the opposite direction to that of upper layer. But as Stommel says in his paper and in his book "Gulf Stream", this meander theory presented here is not complete or proven, but merely suggestive of a type of wave motion which may possibly dominate the dynamics of meanders.

In this paper we also did not include lateral boundaries to the Stream. But to apply this treatment to the true Stream, we should take account of this fact rather than the depth of ocean or lower layer currents.

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