DYNAMIC INSTABILITY OF DIVERGENT FLOW IN A TWO-LAYER OCEAN

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Abstract

This is a preliminary paper for stability of a divergent flow. A very simple model is considered which shows the most crude approximation of the currents structure in the ocean. The effect of horizontal divergence for instability of laminar flow is studied and the results are compared with the results of non-divergent flow.

1. Introduction

In the previous paper (S. Hikosaka, 1960), the extension of Stommel's theory (1953) for Gulf Stream Meanders has been made by the present author, and according to his results the unstable waves may occur under the criterion looser than that of Stommel if the lower layer current flows in the opposite direction to the upper layer current or if the depth of ocean is limitted. However, neither Stommel nor the present author have treated the lateral boundaries to the stream. Haurwitz and Panofsky (1950) showed the existence of unstable waves as a result of shearing instability in a non-divergent flow. But Neumann's analysis (1948) suggests that the vertical structure of the current is of considerable importance for the dynamics of the stream. So, if the horizontal divergence of the current in a two-layer ocean is taken account of, their results will be modified.

In this paper, for the first approximation the ocean is supposed to be homogeneous two layers, and the shearing instability of divergent flow has been studied.

2. Foundamental Perturbation Equations

In the undisturbed state, a steady current U flows in the x-direction within a region with the width of $2d$ in the upper layer of thickness D. And the lower layer is very deep, and hence the horizontal pressure gradients vanish in it at all times. The density difference of two layers is $\Delta\rho=\rho'-\rho$. Under these conditions, the undisturbed current is governed by

$$fU=-g'\frac{\partial D}{\partial y}, \quad \text{where} \quad g'=g\frac{\Delta\rho}{\rho'}. \tag{1}$$

We now suppose that there are small perturbations u, v in the velocity components, and h in the elevation of the free surface. The perturbation equations become

$$\left\{\begin{array}{ll}\left(\dfrac{\partial}{\partial t}+U\dfrac{\partial}{\partial x}\right)u-fv=-g\dfrac{\partial h}{\partial x}, & (2)\\ \left(\dfrac{\partial}{\partial t}+U\dfrac{\partial}{\partial x}\right)v+fu=-g\dfrac{\partial h}{\partial y}, & (3)\\ \left(\dfrac{\partial}{\partial t}+U\dfrac{\partial}{\partial x}\right)\dfrac{\rho'}{\Delta\rho}h+v\dfrac{\partial D}{\partial y}+D\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)=0. & (4)\end{array}\right.$$

If the perturbations are all in the form $e^{\alpha y}e^{ik(x-ct)}$ we obtain the following relation between α and c from the above equations (2), (3) and (4)

$$gD(U-c)\alpha^2-f\frac{\rho}{\Delta\rho}U(U-c)\alpha+\kappa^2\frac{\rho}{\Delta\rho}(U-c)^3$$
$$+f^2\frac{\rho}{\Delta\rho}U-f^2\frac{\rho}{\Delta\rho}(U-c)-gD\kappa^2(U-c)=0 \qquad (5)$$

In this equation we assume $D\to\infty$ then $\alpha^2=\kappa^2$, which Haurwitz and Panofsky obtained in a non-divergent flow. And if we put $\alpha=0$ (that is, the perturbations are all independent of y) we obtain the following frequency equation that is the same as Stommel's,

$$\kappa^2U^2(1-p)^3-\left[f^2+g\kappa^2\frac{\Delta\rho}{\rho}D\right](1-p)+f^2=0,$$

where $p=c/U$.

3. Numerical Calculations

1) Case I.

First, we consider the ocean which the current flows in the x-direction with the constant velocity U within the region $d\geqq y\geqq -d$ in the upper layer (region II) and in the regions of (I) and (III), which is extended to semi-infinity, $U=0$. (See Fig. 1).

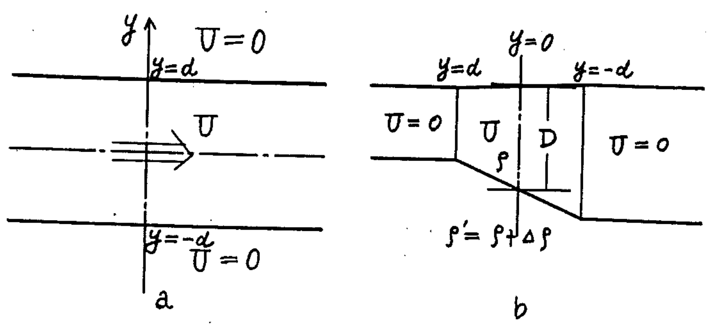

Fig. 1. Model of the Ocean. (Case I.)
a: Horizontal Cross Stream.
b: Vertical Structure.

In this case, we can obtain the frequency equation (6) by taking into account of the internal boundary conditions, that is pressure is continuous, at $y=\pm d$,

$$\{e^{(\alpha_2-\alpha_2')d}-e^{-(\alpha_2-\alpha_2')d}\}$$
$$\times[-b^2\{p^4+(1-p)^4\}+a^2b^2p^2(1-p)^2\{p^2+(1-p)^2\}+a^2p(1-2p)\{p+(1-p)^2\}]$$
$$=e^{(\alpha_2-\alpha_2')d}+e^{-(\alpha_2-\alpha_2')d}\}\times\frac{U^2}{f^2}p^2(1-p)^2\alpha_1(\alpha_2-\alpha_2'), \tag{6}$$

where $\alpha_1=\frac{f}{U}a\sqrt{1+\frac{b^2}{a^2}-b^2p^2}$,

$$\alpha_2-\alpha_2'=\frac{f}{U}a\sqrt{a^2+4+4\frac{b^2}{a^2}-4b^2(1-p)^2-4\frac{1}{1-p}},$$

and $p=c/U,\quad a^2=\frac{U^2\rho}{gD\Delta\rho},\quad b^2=\frac{\kappa^2U^2}{f^2}.$

For non-divergent flow, we can put $a^2=0$, then the equation (6) becomes to

$$(e^{2\kappa d}-e^{-2\kappa d})\{-p^4+(1-p)^4\}=(e^{2\kappa d}+e^{-2\kappa d})\times 2p^2(1-p)^2.$$

The above equation is reduced to

$$p=\frac{1+i\sqrt{\delta\pm\sqrt{\delta^2-1}}}{1+\delta\pm\sqrt{\delta^2-1}} \tag{7}$$

where $\delta=\coth 2\kappa d$.

From the equation (7) we can see that for non-divergent flow the constant δ (this means that the ratio of the width of the current to the wave length is constant) gives the constant velocity of propagation and constant instability for constant current. Therefore we made numerical calculations assuming the several velocities of propagation from the frequency equation (6). Fig. 2 shows the phase velocity and instablility for κd in the case of non-divergent flow. Fig. 3 is the results of numerical calculations for $a^2=0.4$ and shows the relation between the width of current and the length of the

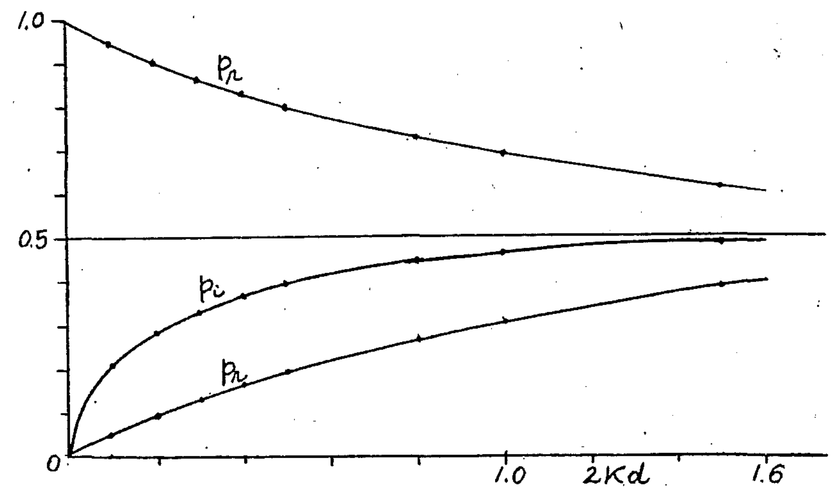

Fig. 2. Phase velocity and instability as a function of κd (Non-divergent flow)

wave which propagates with constant phase velocity. It is seen from Fig. 3 that the horizontal divergence is effective for the waves with long wave-length and the shorter the wave is, the lesser the effect is. Also wa can see that if we assume the velocity of propagation is constant, the instability becomes smaller and the width of current becomes larger as the wave length becomes larger and in the extreme case the width of current is infinity the instability is zero and b^2 (related to the wave number) is 0.09, 0.043 and 0.021 for $a^2=0.4$ (In this case, we can not expect any unstable waves in Stommel's model because of $a^2<1$.) and for $p_r=0.2$, 0.1 and 0.05 respectively. Whereas we get b^2 of 0.135, 0.065 and 0.033 from Stommel's equation and these are about 3/2 times of our results. These discrepancies might be from the boundary conditions at the horizontal interfaces. And for the smaller values than these critical values of b there is no unstable wave with the constant phase velocity. Fig. 4 shows the relation curves of p_i (related to instability) and b (related to wave length). As shown in Fig. 3, for a non-divergent flow we can expect the unstable waves with constant p_r for any wave length, but for a divergent flow we can not expect them unless the wave length is shorter than the value of b for which d is infinity. To the author, this fact is uncertain and he conceives it is due to the assumption that the depth of upper layer D is constant as Stommel did. But as a matter of

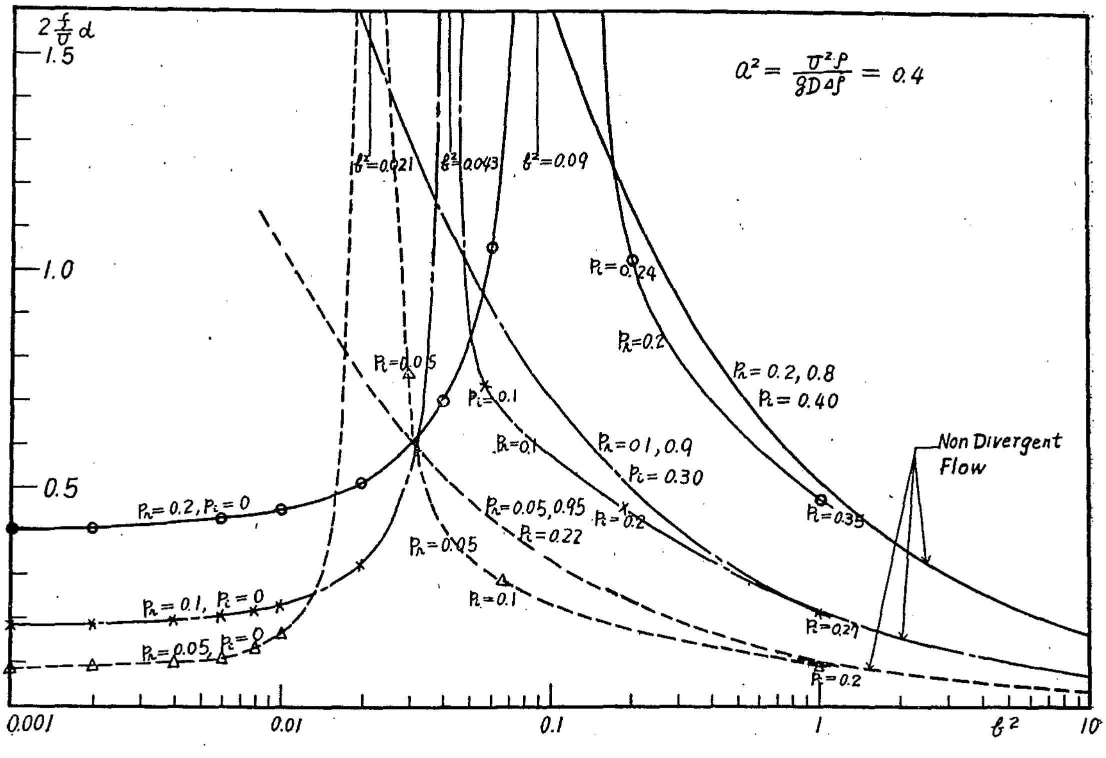

Fig. 3. $b\left(=\frac{\kappa U}{f}\right)\sim 2\frac{U}{f}d$ relation curves for ratios of velocity of propagation to steady current velocity $(p_r=C_r/U)$.

fact, we should consider D varies linearly across the stream since the steady current U is supposed to be constant. So, as the width of current becomes larger, the upper layer becomes thicker.

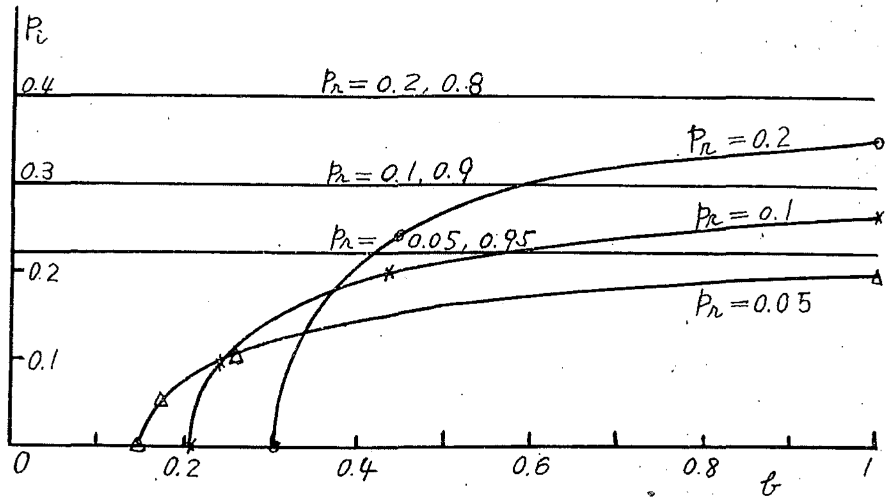

Fig. 4. Relation between p_i and b. ($a^2=0.4$)

2) Case II

We next consider of the ocean which has a rigid boundary (coast). As shown in Fig. 5, the steady current U with the width of $2d_1$ flows apart from the coast at a distance of d_2 (to the center of the current) and on either side of the current rigion there exists no current. In this case we finally get the following frequency equation (8) instead of the equation (6).

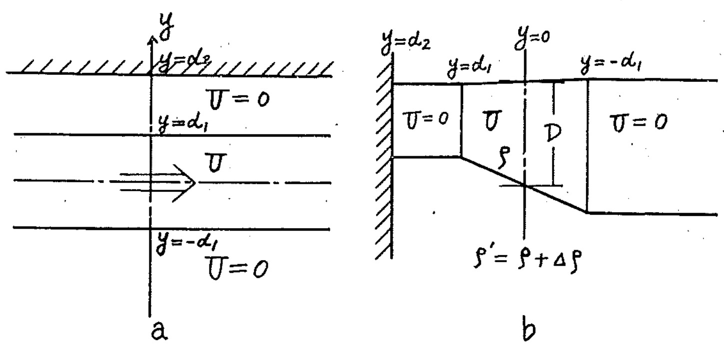

Fig. 5. Model of the Ocean. (Case II.)
a: Horizotnal cross Stream.
b: Vertical Structure.

$$
\begin{aligned}
&[e^{(\alpha_2-\alpha')d_1}-e^{-(\alpha_2-\alpha_2')d_1}]\times\Big\{[-b^2\{p^4+(1-p)^4\}+a^2b^2p^2(1-p)^2\{p^2+(1-p)^2\} \\
&\quad +a^2p(1-2p)\{p+(1-p)^2\}](1-e^{-2\alpha_1(d_2-d_1)}) \\
&\quad +\Bigg[-2p\{1-a^2(1-p)^2\}\frac{pU\alpha_1}{f}+p^2(1-b^2p^2)\{1-a^2(1-p)^2\}\frac{\dfrac{2pU\alpha_1}{f}}{1+\dfrac{pU\alpha_1}{f}}
\end{aligned}
$$

$$+p(1-p)\{2-a^2(1-p)\}\frac{pU\alpha_1}{f}\Big]e^{-\alpha_1(d_2-d_1)}\Big\}$$
$$=[e^{(\alpha_2-\alpha_2')d_1}+e^{-(\alpha_2-\alpha_2')d_1}]\times\frac{U^2}{f^2}p^2(1-p)^2\alpha_1(\alpha_2-\alpha_2'), \tag{8}$$

where α_1 and $\alpha_2-\alpha_2'$ are the same as those of case I.
If non-divergent flow is considered, putting $a^2=0$,

$$p=\frac{1+i\sqrt{\dfrac{\delta\pm\sqrt{\delta^2+\gamma^2-1}}{1-\gamma}}}{1+\dfrac{\delta\pm\sqrt{\delta^2+\gamma^2-1}}{1-\gamma}}, \tag{9}$$

where $\delta=coth\ 2\kappa d_1$ and $\gamma=e^{-2k(d_2-d_1)}$.

If we take $d_2\to\infty$ in the above equations (8) and (9), they become to the same equations as those in case I. Especially, when the current flows close to the coast $d_2=d_1$, $\gamma=1$ and then we get

$$p=\frac{1+i\sqrt{1/\delta}}{1+1/\delta}. \tag{10}$$

Even if d or d_1 tends to infinity in the equation (7) or (10), p_i does not become 0. This means the internal boundary exists at infinity. So, if we want to or should obtain the stable waves when the width of the current is infinity, we must assume the existance of the rigid boundaries in the regions of either side of the stream. We have not made numerical calculations in case II yet, and the results of this case will be shown on the following volume.

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