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DETERMINATION OF RELATIVE GEODETIC POSITION FROM SIMULTANEOUS OBSERVATIONS OF ARTIFICIAL SATELLITES

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Abstract

Three geometrical methods for determination of relative geodetic position using artificial satellites, that is, the simultaneous method, the trailing method, and the simultaneous-trailing method are briefly outlined. The latter two of them are of very efficiency, because they do not always require precise timing device at all stations.

The procedure of reduction for these geometrical methods is described in detail. Since the observation equations can be derived to so-called "position plane", we can solve them together by assigning a weight to each equation.

Formulae for weighting of position planes and corrections for phase effect of satellite are given in appendices.

1. Introduction

The problem on the geometric connections between points on the surface of the earth by observations of artificial satellites has been investigated by many authors. For the use of the passive satellites, simultaneous method, which is also called as the method of synthetic observations, is widely employed in the world and a method so-called trailing method was proposed by Hirose (1963) and is used frequently in Japan. The latter does not necessitate timing device except at one of base stations, while the former requires precise timing device at each station.

In a previous paper (Yamazaki, 1968), the author discussed the relative advantages and disadvantages of these two methods of satellite triangulation, and pointed out that the trailing method has advantages in efficiency and in cost, for the reason that ordi-

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nary astrocameras may be used. In the same paper, he also said that the trailing method needs to utilize satellites of proper inclination. Afterwards, from some experimental observations, the author found that a method which combines these two methods is very effective, even if we could not use but high inclination satellites. We shall call this method as " simultaneous-trailing method " hereafter.

In the present paper, the methods of determining relative position by the above mentioned methods of observation are discussed. And formulae for weighting the observation equations and corrections for phase effect of satellite are given in appendices.

2. Out line of three methods of satellite triangulation

1) Trailing method

In Fig. 1 a, A is a base station having a camera with a timing device, B is another base station having a simple camera on a equatorial mounting without any timing device, X is an unknown station having the same camera as the station B, and S_0 is the satellite position at a time t. We want to determine the position of X geodetically.

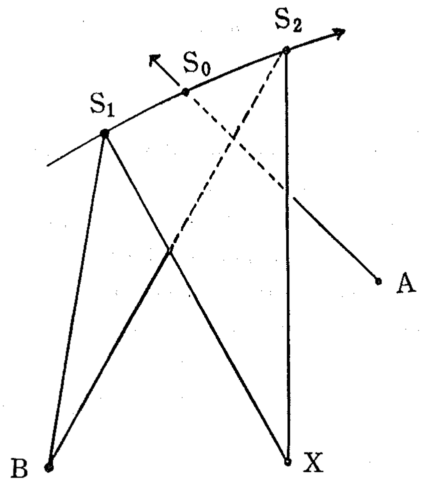

Fig. 1 a Trailing Method.

Suppose that the satellite is photographed at A, B and X at about the same time. The photographic plate taken at A shows the dotted (or chopped) images of the satellite produced by the timing device and those taken at B and X show the trail image of satellite around t.

The direction $\overrightarrow{AS_0}$ of S_0 as seen from A is obtained by interpolation from the dotted images to which time is tagged on the photographic plate taken at this station, with reference to background stars. As is clearly seen from the figure, S_0 is the point of intersection of satellite's trajectory $\widehat{S_1S_2}$ in space and the plane AS_0B defined by A, B and $\overrightarrow{AS_0}$. Accordingly, the image point corresponding to the satellite position at t on the photographic plate taken at B is given as the intersection of the great circle containing the projection of $\widehat{AS_0}$ as seen from B on the celestial sphere with the trail image, and the direction $\overrightarrow{BS_0}$ are determined with reference to the background stars in the same way as $\overrightarrow{AS_0}$.

Since the coordinates of A and B are known, the space coordinates of S_0 are fixed as the intersection of two straight lines AS_0 and BS_0 which have the directions $\overrightarrow{AS_0}$ and $\overrightarrow{BS_0}$, and pass through A and B, respectively. The unknown point X should be on a plane which is defined by S_0 and the projection $\widehat{S_1S_2}$ of the satellite trail on the celestial sphere as seen from X.

Hence, the coordinates of X are given by the intersection of such three or more planes.

2) Simultaneous method

In Fig. 1 b, as before, A and B are two base stations, X is unknown stations and S_0 is satellite position at time t. Each of these stations are equipped with precise timing device.

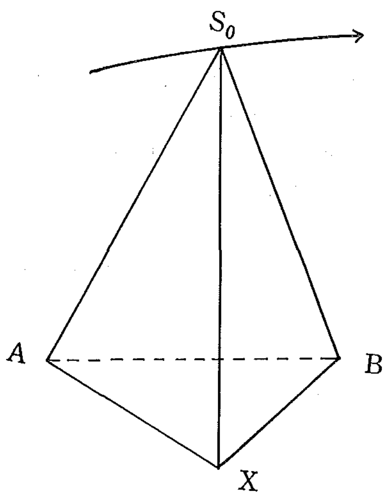

Fig. 1 b Simultaneous Method.

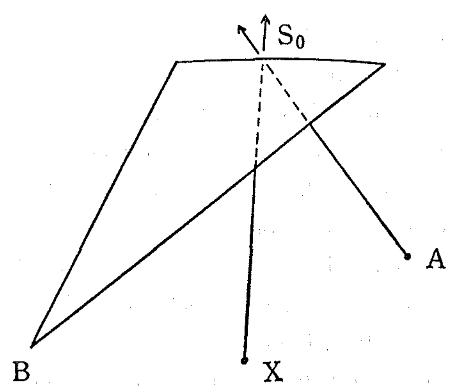

Fig. 1 c Simultaneous-Trailing Method.

The directions $\overrightarrow{AS_0}$, $\overrightarrow{BS_0}$ and $\overrightarrow{XS_0}$ as seen from A, B and X can be derived, with reference to their background stars. The straight lines AS_0 passing through A in the direction of $\overrightarrow{AS_0}$ and the direction $\overrightarrow{XS_0}$ define a plane AS_0X, called as "position plane" hereafter. Similarly, the straight lines BS_0 and XS_0 define another position plane BS_0X. As X is on the intersection $\overline{XS_0}$ of the position planes AS_0X and BS_0X, we can fix the position of X as the point of intersection of $\overline{XS_0}$ and another position plane obtained from the similar simultaneous observation in a different position.

So far, for the sake of brevity, we have assumed that the positions of A and B are known. However, the position of B relative to A can be fixed by two simultaneous observations made at A and B, if only the distance between A and B is previously given. Accordingly, the simultaneous method makes possible the determination of the absolute position of stations independently to existing geodetic system covering A and B.

3) Simultaneous-Trailing method

In Fig. 1 c, A is a base station having a camera with a precise timing device, B is another base station having a simple camera on a equatorial mounting without any timing device, X is an unknown station having the same camera as A, and S_0 is the satellite position in space at time t.

In this case, the space coordinates of S_0 are derived from the observations made at stations A and B, in the same way as the case of the trailing method.

As X should be on the straight line $\overline{S_0X}$ which has the direction of the satellite as seen from X at time t and passes S_0, $\overline{S_0X}$ is a kind of position line of X. Obviously, the position line is given by only one set of simultaneous observation at three stations, and X is fixed as the intersection of such two position lines.

As we have seen in the preceding sub-sections, the observation equations in the trailing-and the simultaneous methods are given by the same form expressed by the position plane. Accordingly, in this case, too, it will be a great convenience to express the observation equations by two position planes which their intersection denotes position line $\overline{S_0X}$.

As can easily be seen, one of these position planes AS_0X can be directly derived from the pair of observations made at A and X in the same way as the case of the simultaneous method. On the other hand, another position plane BS_0X is obtained by combining the straight line $\overline{BS_0}$ fixed by the coordinates of S_0 and B with the observed direction $\overrightarrow{XS_0}$.

This simultaneous-trailing method is available even if we have high inclination satellites alone, for which the trailing method is hardly be applied. Further, we can expect to get more succesful observations than the case of the simultaneous method, because much more pairs of stations having ordinary astrocameras without timing device can be treated as the base stations.

3. Reference coordinate systems

Prior to the description of the reduction technique, the definitions of reference coordinate systems which will be used throughout this paper are given in this section.

The first coordinate system (X, Y, Z) is defined by the actual axis of rotation of the earth (true pole) as the Z-axis, and the true vernal equinox as the X-axis and the point on the equator, the right ascension of which is 6^h, as the Y-axis and centered at the center of gravity of the earth. In reality, this coordinate system is replaced by a system defined by the apparent places of the background stars belonging to a catalogue. We shall call this system as the sidereal coordinate system according to Veis (1963).

The second coordinate system (U, V, W) is obtained by rotating the first system about Z-axis so that XZ-plane coincides with the true Greenwich meridian. This system is related to the first system by the following expression:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{1}$$

or

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\begin{pmatrix} \cos H & -\sin H & 0 \\ \sin H & \cos H & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{2}$$

where H is the true sidereal time calculated from observation time in UT1 by using the Ephemerides. We shall call this system as the true terrestrial rectangular coordinate system.

The second system refers evidently to the instanteneous pole and is not fixed with respect to the earth surface. In this system, as the coordinates of stations vary as time goes on, it is inconvenient to solve the problems of position determination from observations spread over a long time interval. Accordingly, we shall introduce the third coordinate system (U_0, V_0, W_0) defined by Veis (1963) as follows. He call this system as the ideal terrestrial rectangular coordinate system.

The origin is at the center of gravity of the earth analogous to the 1st and 2nd system, and oriented so that W_0-axis is directed towards the mean north pole as defined by the International Latitude Service. The U_0-W_0 plane is parallel to the mean Greenwich meridian as defined by the Bureau Internationale de l'Houre.——At the XIIIth General Assembly of IAU in Prague (1967), a resolution was adopted that the origin of the north pole is defined by the mean pole 1900–1905 from 1967, which is called as the Conventional International Origin (CIO). Thus the above-defined mean north pole and the mean meridian are referred to the CIO, hereafter.

The actual motion of the true pole, defined by the instantaneous axis of the rotation of the earth, is determined by the International Polar Motion Service, which gives the position of true pole in terms of the angular coordinates (x, y) of the instantaneous pole with respect to the CIO.

The third system is related to the second system by the expression

$$\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & -y \\ -x & y & 1 \end{pmatrix}\cdot\begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{3}$$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}=\begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & y \\ x & -y & 1 \end{pmatrix}\cdot\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}. \tag{4}$$

From (1) and (3), therefore, the relation between the first and third systems is given by the expressions (Veis, 1963),

$$\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}=\begin{pmatrix} \cos H & \sin H & x \\ -\sin H & \cos H & -y \\ -x\cos H-y\sin H & -x\sin H+y\cos H & 1 \end{pmatrix}\cdot\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{5}$$

or

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\begin{pmatrix} \cos H & -\sin H & -x\cos H-y\sin H \\ \sin H & \cos H & -x\sin H+y\cos H \\ x & -y & 1 \end{pmatrix}\cdot\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}. \tag{6}$$

In the reduction process of the satellite triangulation, the coordinates of satellites are given by the (XYZ) system, and the stations by the $(U_0 V_0 W_0)$ system. They are related by the formula (5) or (6).

In order to get the three-dimensional coordinates of the stations in the terrestrial rectangular coordinate system, namely, independently of the reference ellipsoid, it is sufficient to adopt the above coordinate systems. However for those practical triangulation problems such as linking of an isolated island with a triangulation net of main land, the positions of the stations are usually given by the geodetic coordinate system (λ, φ, h).

Unless the reference ellipsoid is properly oriented in the terrestrial rectangular coordinate system, it is impossible in principle to transform the geodetic coordinates into the terrestrial rectangular coordinates.

So far as our purposes are limited to the relative position determination, we can take the following well-known relation between the geodetic coordinates and the terrestrial rectangular coordinates, within the accuracy of the triangulation nets.

$$\begin{aligned} U_0 &= (N+h)\cos\varphi\cos\lambda\,, \\ V_0 &= (N+h)\cos\varphi\sin\lambda\,, \\ W_0 &= [N(1-e^2)+h]\sin\varphi\,, \end{aligned} \tag{7}$$

and

$$N = a\,(1-e^2\sin^2\varphi)^{-1/2}\,,$$

where a and e mean the equatorial radius and the eccentricity of the reference ellipsoid, respectively.

4. Plate reduction

The method of plate constants is adopted to derive the celestial coordinates of the satellite from the plate.

We shall take the reference stars having magnitude $(8^m \sim 9^m)$ as many as possible in the vicinity of the satellite images concerned (within a field of about 3° diameter).

Let (x_i, y_i) and (x_j, y_j) respectively denote the measured coordinates of reference stars and satellite's chopped images (or subdivided points of satellite's trailed image in the case without timing device).

As the celestial coordinates of the reference stars, we will adopt the apparent right ascension and declination α_i, δ_i which are computed from a star catalogue and the ephemeris, because the adoption of the mean place complicates a situation, especially in the trailing method.

As well known, the relations between the celestial coordinates α_i, δ_i and the standard coordinates ξ_i, η_i are shown by

$$\begin{aligned} \cos\delta_i\cos\alpha_i &= l_i\,, \\ \cos\delta_i\sin\alpha_i &= m_i\,, \\ \sin\delta_i &= n_i\,, \end{aligned} \tag{8}$$

$$\begin{aligned}
&-\sin\alpha_0\cdot l_i+\cos\alpha_0\cdot m_i =\tilde{\xi}_i\,,\\
&-\sin\delta_0\cdot\cos\alpha_0\cdot l_i-\sin\delta_0\cdot\sin\alpha_0\cdot m_i+\cos\delta_0\cdot n_i=\tilde{\eta}_i\,,\\
&\cos\delta_0\cdot\cos\alpha_0\cdot l_i+\cos\delta_0\cdot\sin\alpha_0\cdot m_i+\sin\delta_0\cdot n_i=\zeta_i\,,\\
&\tilde{\xi}_i/\zeta_i=\xi_i\,,\\
&\tilde{\eta}_i/\zeta_i=\eta_i\,,
\end{aligned}\tag{9}$$

where α_0 and δ_0 mean the right ascension and declination of the plate center. (Generally in a conventional satellite camera, a guiding telescope is not used to keep a star on the plate center, so we only know the approximate values of α_0, δ_0 at first. If we take account of the higher order terms for the plate constants as shown later, the errors resulting from such improper orientation of the plate center are almost negligible. However, even when we must solve the problems by the method of linear plate constants, due to insufficient number of the reference stars, we can succesively derive the approximate values of α_0, δ_0, provided that the measured coordinates of the plate center are known.)

So far as the higher order terms which originate from differential refraction, decentering error, tilting error and camera distorsion etc., can be neglected, the general relations between the standard coordinates and the measured coordinates are given with sufficient accuracy by the following linear expressions (Turner's Method).

$$\begin{aligned}
&\xi_i=ax_i+by_i+c\,,\\
&\eta_i=dx_i+ey_i+f, \qquad i=1,\cdots,n\,.
\end{aligned}\tag{10}$$

The coefficients $a, b, c, \cdots, f$ are called the plate constants.

If the zenith distance is larger than 60° and the camera distorsion cannot be neglected, the second or higher terms must be taken into account. In this connection, many different formulae have been derived so far.

The auther presently use the following formula given by W. E. Good et. al. (1962).

$$\begin{aligned}
&\xi_i=ax_i+by_i+c+dx_iy_i+ex_i^2+f(x_i^2+y_i^2)x_i\,,\\
&\eta_i=gx_i+hy_i+i+jx_iy_i+ky_i^2+l(x_i^2+y_i^2)y_i\,,\\
&i=1,\cdots,n\,.
\end{aligned}\tag{11}$$

Here, it should be noted that (11) were derived from the assumption that x-axis and ξ-axis are parallel to each other. In use of a comparator without a turn table, we cannot set the x-axis of the comparator parallel to ξ-axis, in general.

In order to solve this problem, the author proposed the following method.

We first determine the plate constants $a, b, c, \cdots, f$ in (10) by applying the method of least squares.

Let θ be the angle between ξ- and x-axis, and κ_1, κ_2 the scale factors in ξ- and η-directions, respectively, then the following relations (Muller 1964) are held between (a, b, d, e) and $(\kappa_1, \kappa_2, \theta)$:

$$\begin{aligned}
&a=\kappa_1\cos\theta\,, \qquad d=\kappa_1\sin\theta\,,\\
&b=\kappa_2\sin\theta\,, \qquad e=-\kappa_2\cos\theta\,,
\end{aligned}\tag{12}$$

and θ may be calculated from the values of a, b, d, e by using the formulae

$$\tan\theta=\frac{d}{a}\,, \qquad \text{or} \qquad -\frac{b}{e}\,. \tag{13}$$

Though the two values of θ derived from (13) do not strictly agree with each other, we can approximately take their mean value in general cases.

Transform the measured coordinates (x_i, y_i) into the new coordinates $(\bar{x}_i, \bar{y}_i)$ which are given by the following relations.

$$\begin{aligned} \bar{x}_i &= x_i \cos\theta + y_i \sin\theta \,, \\ \bar{y}_i &= x_i \sin\theta - y_i \cos\theta \,. \end{aligned} \tag{14}$$

Obviously, the above equations show that $\bar{x}$-axis is approximately parallel to ξ-axis.

By using the values of $(\bar{x}_i, \bar{y}_i)$ in place of (x_i, y_i), we can solve (11) by the method of least squares.

Here, it is also to be noted that at least three stars are needed for the solution, because (10) have six unknown plate constants. On the other hand, as (11) have twelve unknown plate constants, six stars are needed at least.

The standard coordinates of the satellite's images (ξ_j, η_j) can be calculated from (10) or (11) for the corresponding measured coordinates (x_j, y_j) by using the plate constants just determined above.

Finally, the celestial coordinates of the satellite (α_j, δ_j) are given by

$$\begin{aligned} \tilde{\zeta}_j &= 1/\sqrt{\xi_j^2+\eta_j^2+1} \,, \\ \tilde{\eta}_j &= \eta_j \cdot \tilde{\zeta}_j \,, \\ \tilde{\xi}_j &= \xi_j \cdot \tilde{\zeta}_j \,, \\ l_j &= \cos\alpha_0 (\cos\delta_0 \cdot \tilde{\zeta}_j - \sin\delta_0 \cdot \tilde{\eta}_j) - \sin\alpha_0 \cdot \tilde{\xi}_j \,, \\ m_j &= \cos\alpha_0 \cdot \tilde{\xi}_j + \sin\alpha_0 (\cos\delta_0 \cdot \tilde{\zeta}_j - \sin\delta_0 \cdot \tilde{\eta}_j) \,, \\ n_j &= \cos\delta_0 \cdot \tilde{\eta}_j + \sin\delta_0 \cdot \tilde{\zeta}_j \,, \\ \sin\delta_j &= n_j \,, \\ \tan\alpha_j &= m_j / l_j \,. \end{aligned} \tag{15}$$

5. Smoothing of satellite's positions

In satellite astrometry, it is to be especially noted that the duration of the exposure for background stars and a satellite may differ from each other very much and occur at widely different times. As a result, the image motion of satellite caused by the atmospheric disturbances in the neighbourhood of camera may affect the mean position of the trailed or dotted images of the satellite, while the star images are averaged about their true positions in case of using an equatorial mounting camera Accordingly, we must be careful to smooth the observed positions of satellite.

The satellite cameras of the Hydrographic Department of Japan are equipped with a special timing device having a travelling slit, which is designed to make three groups of eight dot images aparting about three degrees from each other on a plate (Ono 1966).

The practical smoothing technique for the satellite positions observed by this camera is described below. (For the principle of this smoothing method, the reader should refer to the previous paper of the author (1968)).

Now the standard coordinates of the satellite images at time t_j, ξ_j, η_j can be calculated from the measured coordinates by using (10) or (11). The standard coordinates ξ_j, η_j are transformed into the new standard coordinates $\bar{\xi}_j$, $\bar{\eta}_j$ which are defined by

$$\bar{\xi}_j=\xi_j \cos\theta+\eta_j \sin\theta\,, \tag{16}$$
$$\bar{\eta}_j=-\xi_j \sin\theta+\eta_j \cos\theta\,, \qquad j=1, \cdots, 8 \text{ for each group,}$$

and

$$\tan\theta\equiv(\eta_8-\eta_1)/(\xi_8-\xi_1)\,.$$

The $\bar{\xi}$-axis is clearly taken nearly parallel to the direction of the satellite trail.——We shall call them as the modified standard coordinates, hereafter.

Assuming a relation between $\bar{\xi}_j$ and $\bar{\eta}_j$,

$$\bar{\eta}_j=E\cdot\bar{\xi}_j^2+F\cdot\bar{\xi}_j+G\,, \tag{17}$$

we determine the coefficients E, F, and G by the least squares procedure.

As obviously seen from the definition of the modified standard coordinates, the variation of $\bar{\eta}$ with respect to time is very small. Therefore, it is sufficient to assume merely the following relation between $\bar{\xi}_j$ and t_j, in order to smooth the satellite positions with respect to time.

$$\bar{\xi}_j=E'\cdot t_j^2+F'\cdot t_j+G'\,. \tag{18}$$

The coefficients E', F' and G', also, can be determined by the method of least squares.

Let us, now, find the smoothed satellite position at any time t between t_1 and t_8.

By substituting the value of t in (18), we obtain $\bar{\xi}$ corresponding to t. Similarly, $\bar{\eta}$ corresponding to $\bar{\xi}$ is given by (17). The transformations of the standard coordinates $\bar{\xi}$, $\bar{\eta}$ into the celestial coordinates α, δ are made through (16), (15), with the aid of the coordinates of plate center α_0, δ_0 already known.

The above procedure will be used only for $t_1<t<t_8$. When the satellite image at time t falls between each group of eight dot images, the procedure will not be so applicable due to extrapolation.

In this case, it may be well to apply Hirose's preliminary orbital smoothing method (1962) to the three smoothed positions of satellite which are obtained by the above procedure at the central times of exposures for each group t_{I}, t_{II} and t_{III}.

6. Reduction of trailing method

1) Determination of the direction of satellite as seen from B at time t

Let us first derive the equation of the great circle that expresses the plane ABS_0 in Fig. 1a in terms of the modified standard coordinates from the plate taken at station B. Hereupon, it is to be noted that generally this great circle cannot be expressed by a linear form with respect to the measured coordinates, if we want take account of the higher terms of the plate constants.

Before going into the problem, we obtain the celestial coordinates (α_A, δ_A) of the satellite position as seen from A at time t by the procedure of the plate reduction described in the foregoing section, and calculate the direction cosines l_A, m_A, n_A corresponding to (α_A, δ_A) from (8). And we can also calculate the direction cosines l_{BA}, m_{BA}, n_{BA} of B as seen from A given by the formulae

$$l_{BA}=(x_B-x_A)/\varDelta_{BA}\,,\qquad (19)$$
$$m_{BA}=(y_B-y_A)/\varDelta_{BA}\,,$$
$$n_{BA}=(z_B-z_A)/\varDelta_{BA}\,,$$

with

$$\varDelta_{BA}=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}.$$

Here, x_A, y_A, z_A ; x_B, y_B, z_B respectively mean the geocentric rectangular coordinates of stations A and B in the sidereal coordinate system (X, Y, Z) derived from the relations (7) and (6) for their given geodetic coordinates.——According to the previous paper (Yamazaki, 1968), we shall use the refraction heights instead of the geodetic heights of stations through this paper.*

As already stated the plane ABS_0 is defined by the direction $\overrightarrow{AS_0}$ and the straight line $\overline{AB}$. Thus the equation of great circle is given by

$$\begin{vmatrix} l & m & n \\ l_A & m_A & n_A \\ l_{BA} & m_{BA} & n_{BA} \end{vmatrix}=0\,,\qquad (20)$$

where (l, m, n) mean the direction cosines of an arbitrary vector on the plane ABS_0 (Fig. 1a).

We rewrite (21)

$$l\cdot A+m\cdot B+n\cdot C=0\,,\qquad (21)$$

with

$$\mathrm{A}=\begin{vmatrix} m_A & n_A \\ m_{BA} & n_{BA} \end{vmatrix},\quad B=\begin{vmatrix} n_A & l_A \\ n_{BA} & l_{BA} \end{vmatrix},\quad C=\begin{vmatrix} l_A & m_A \\ l_{BA} & m_{BA} \end{vmatrix}.$$

The trasformation of (l, m, n) into the corresponding standard coordinates (ξ, η) is given by the following equations derived from (9) or (15):

$$l=\zeta\{(\cos\delta_0-\sin\delta_0\cdot\eta)\cos\alpha_0-\xi\cdot\sin\alpha_0\}\,,$$
$$m=\zeta\{(\cos\delta_0-\sin\delta_0\cdot\eta)\sin\alpha_0+\xi\cdot\cos\alpha_0\}\,,\qquad (22)$$
$$n=\zeta\{\cos\delta_0\cdot\eta+\sin\delta_0\}\,.$$

On the other hand, the relation between (ξ, η) and the modified standard coordinates $(\bar{\xi}, \bar{\eta})$ is given by (16).

Thus, by means of (16) and (22), the equation of the great circle (20) can be transformed into the following equations in terms of the modified standard coordinates:

$$p\cdot\bar{\xi}+q\cdot\bar{\eta}+r=0\,,\qquad (23)$$

where

$$p=(B\cos\alpha_0-A\sin\alpha_0)\cos\theta+C\cos\delta_0-A\sin\delta_0\cos\alpha_0-B\sin\delta_0\sin\alpha_0)\sin\theta\,,$$
$$q=-(B\cos\alpha_0-A\sin\alpha_0)\sin\theta+(C\cos\delta_0-A\sin\delta_0\cos\alpha_0-B\sin\delta_0\sin\alpha_0)\cos\theta\,,$$
$$r=A\cos\delta_0\cos\alpha_0+B\cos\delta_0\sin\alpha_0+C\sin\delta_0\,.$$

We have already seen that the direction of satellite as seen from B at time t is given by the intersection point of this great circle with the satellite trail.

Hence, our next step is to obtain the equation corresponding to the satellite trail on the plate taken at station B (without timing device).

* If we want to treat with the corrections of the parallactic refractions, in the trailing method, we must apply the corrections for all subdivided points on the satellite trail.

Let x_j, y_j be the measured coordinates of the subdivided points (e.g. with a space of about 2 mm) of the trailed image. The (x_j, y_j), by means of (10) or (11) and (16), are transformed into the corresponding (ξ_j, η_j), with the aid of the plate constants determined by the procedure of the section 4.

If we take enough short part of trail, this part can be approximated by the following formula to sufficient accuracy.

$$\eta_j = E_B \cdot \xi_j^2 + F_B \cdot \xi_j + G_B\,, \tag{24}$$

coefficients of right hand side can be also obtained by the method of least squares. Thus, the coordinates of intersection point of the great circle with the satellite trail (ξ_B, η_B) is given by the two equations (23) and (24).

Transformation of (ξ_B, η_B) into (l_B, m_B, n_B), which is defined as direction cosines of satellite as seen from B, can be carried out through (16), (15).

In passing, here we will derive the quantities needed in the later section.

We chose a point on the satellite trail about 0.1° apart from (ξ_B, η_B). The coordinates (ξ'_B, η'_B) of this point are

$$\left.\begin{aligned} \xi'_B &= \xi_B + 0.1 \times \sin 1^\circ\,, \\ \eta'_B &= E_B \cdot \xi'^2_B + F_B \cdot \xi'_B + G_B\,. \end{aligned}\right\} \tag{25}$$

Then, (ξ'_B, η'_B) are transformed into the corresponding direction cosines (l'_B, m_B, n'_B) after the same procedure as described above.

Finally, we derive the quantities L, M, N, which denote the direction cosines of normal vector of the plane BS_1S_2 of Fig. 1 a, from the following equations:

$$\begin{aligned} L &= m_B n'_B - n_B m'_B \\ M &= n_B l'_B - l_B n'_B \\ N &= l_B m'_B - m_B l'_B\,. \end{aligned} \tag{26}$$

2) Determination of geocentric coordinates of satellite

Let α, β and γ be the interior angles $\angle S_0AB$, $\angle ABS_0$ and $\angle BS_0A$ of $\varDelta ABS_0$ in Fig 1 a, respectively. Then we can easily see that they will be given by

$$\begin{aligned} \cos\alpha &= l_A l_{BA} + m_A m_{BA} + n_A n_{BA}\,, \\ \cos\beta &= -(l_B l_{BA} + m_B m_{BA} + n_B n_{BA})\,, \\ \gamma &= 180^\circ - (\alpha + \beta)\,. \end{aligned} \tag{27}$$

And the distance $\varDelta_A$ and $\varDelta_B$ from A and B to the satellite are given by

$$\left.\begin{aligned} \varDelta_A &= \varDelta_{BA} \sin\beta / \sin\gamma\,, \\ \varDelta_B &= \varDelta_{BA} \sin\alpha / \sin\gamma\,. \end{aligned}\right\} \tag{28}$$

Thus the geocentric coordinates of the satellite (x_s, y_s, z_s) can be calculated by

$$\begin{aligned} x_s &= x_A + l_A \varDelta_A = x_B + l_B \varDelta_B\,, \\ y_s &= y_A + m_A \varDelta_A = y_B + m_B \varDelta_B\,, \\ z_s &= z_A + n_A \varDelta_A = z_B + n_B \varDelta_B\,. \end{aligned} \tag{29}$$

It will be noted, here, that the aberration effect, as already stated in the preceding paper (Yamazaki, 1968), can be ignored in the trailing method, so long as the required accuracy is the order of 1″ for satellite position.

3) To derive the position plane

In section 1, the position plane is defined as such one that the coordinates of satellite in space make with the projection of satellite trail on the celestial sphere as seen from the unknown station.

Although the satellites generally draw a curved line in the space, we can take such small segement that can be regarded as a straight line on the satellite trail.

Suppose that we could take such small segement on the satellite trail close to the satellite position at time t. Further, let (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X) be the direction cosines of the both ends of this sedgement as seen from X.

Obviously, X is on the plane defined by the two straight lines which have respectively the direction cosines (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X), and pass through the previously obtained satellite position (x_S, y_S, z_S).

Thus, we can express the equation of the position plane by

$$\begin{vmatrix} x-x_S & y-y_S & z-z_S \\ l'_X & m'_X & n'_X \\ l''_X & m''_X & n''_X \end{vmatrix} = 0 . \tag{30}$$

Now the problem is reduced to finding the direction cosines (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X).

A point corresponding to S_0 is situated somewhere on the trail image but its position is unknown, because the station X is not equipped with a timing device. We now introduce an assumed position X' in place of X (for this assumed position it will be sufficient to know within the order of 1 mile).

Let $\Delta_{X'}$ be the distance between S_0 and X', which is given by

$$\Delta_{X'} = \sqrt{(x_S - x_{X'})^2 + (y_S - y_{X'})^2 + (z_S - z_{X'})^2} , \tag{31}$$

where $(x_{X'}, y_{X'}, z_{X'})$ means the geocentric rectangular coordinates of X' and are calculated by (7), (6) for the geodetic coordinates. (Needless to say, we must use the refraction height for the height of X').

Then, the direction cosines of S_0 as seen from X' are given by

$$\begin{aligned} l^{\circ}_X &= (x_S - x_{X'})/\Delta_{X'} , \\ m^{\circ}_X &= (y_S - y_{X'})/\Delta_{X'} , \\ n^{\circ}_X &= (z_S - z_{X'})/\Delta_{X'} . \end{aligned} \tag{32}$$

Transformation of $(l^{\circ}_X, m^{\circ}_X, n^{\circ}_X)$ into the modified standard coordinates $(\bar{\xi}^{\circ}_X, \bar{\eta}^{\circ}_X)$ is carried out by (9), (16).

Now, let $(\bar{\xi}_j, \bar{\eta}_j)$ be the modified standard coodinates of subdivided points on the satellite trail photographed at X. They can be derived by the same procedure as foregoing section.

In order to obtain a smoothed satellite trail, we also assume the following relation between $\bar{\xi}_j$ and $\bar{\eta}_j$, and determine its coefficients by the method of least squares:

$$\bar{\eta}_j = E_X \cdot \bar{\xi}_j^2 + F_X \cdot \bar{\xi}_j + G_X , \qquad j = 1, 2, \cdots . \tag{33}$$

If the values of the assumed coordinates of X' and the satellite coordinates derived above are free from errors, the point $(\bar{\xi}^{\circ}_X, \bar{\eta}^{\circ}_X)$ should be on the satellite trail and show the true direction of the satellite as seen from X at time t, but it is not the general case. The direction of the satellite as seen from the assumed position will not deviate from the true direction by more than 0.1°, so far as the assumed position is not apart from the true position by one mile or more and we observe the satellites passing through at an altitude higher than 1000 km. Here, it should be noted that we can regard an arc length of 0.1° on the satellite trail as a straight line within the accuracy concerned.

Let a point $(\bar{\xi}'_X, \bar{\eta}'_X)$ on the satellite trail, now, be the nearest point to the point $(\bar{\xi}^{\circ}_X, \bar{\eta}^{\circ}_X)$. From the above reason, then this point will be situated within 0.1° from the true direction of the satellite at time t. On putting $\bar{\xi}'_X = \bar{\xi}^{\circ}_X$, and calculating the value of $\bar{\eta}'_X$ by (33), we can easily find the point $(\bar{\xi}'_X, \bar{\eta}'_X)$, since the $\bar{\xi}$-axis is nearly parallel to the satellite trail.

Next, we take a point $(\bar{\xi}''_X, \bar{\eta}''_X)$ close by the point $(\bar{\xi}'_X, \bar{\eta}'_X)$ on the satellite trail, say, at a distance of 0.1°. In the case of this example, the value of $\bar{\xi}''_X$ is calculated by

$$\bar{\xi}''_X = \bar{\xi}'_X + 0.1 \times \sin 1^{\circ}, \tag{34}$$

and the value of $\bar{\eta}''_X$ is obtained from (33) by substitution $\bar{\xi}''_X$.

It will be self-evident from the above discussion that the points $(\bar{\xi}'_X, \bar{\eta}'_X)$ and $(\bar{\xi}''_X, \bar{\eta}''_X)$ respectively correspond to the directions (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X) in (30).

Finally, transformations of $(\bar{\xi}'_X, \bar{\eta}'_X)$ and $(\bar{\xi}''_X, \bar{\eta}''_X)$ into (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X) are carried out by (16), (15).

4) Observation equations

Equation (30) appears to be solved by three sets of simultaneous observations since it contains only three unknowns. As is clearly seen from (6), however, the variables x, y, and z in (30) are given as a function of time. Accordingly, we cannot solve the equation from the sets of observations made at different times, as it stands so far.

In order to solve (30), we must rewrite the equation by using the ideal terrestrial rectangular coordinates (U_0, V_0, W_0) instead of the sidereal coordinate system (X, Y, Z), as follows:

$$\begin{vmatrix} u_0 - u_{0S} & v_0 - v_{0S} & w_0 - w_{0S} \\ l'_{0X} & m'_{0X} & n'_{0X} \\ l''_{0X} & m''_{0X} & n''_{0X} \end{vmatrix} = 0, \tag{35}$$

where (u_{0S}, v_{0S}, w_{0S}), $(l'_{0X}, m'_{0X}, n'_{0X})$, and $(l''_{0X}, m''_{0X}, n''_{0X})$ correspond to (x_S, y_S, z_S), (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X), respectively, and they can be calculated from (5) by substituting (x_S, y_S, z_S), (l'_X, m'_X, n'_X) and (l''_X, m''_X, n''_X) for (X, Y, Z), respectively.

When we solve (35) practically, we had better introduce, as before, the assumed position. Here it is remarked that cofactors of (u_0, v_0, w_0) are not sufficient in the number of digits, since $(l'_{0X}, m'_{0X}, n'_{0X})$ and $(l''_{0X}, m''_{0X}, n''_{0X})$ are nearly parallel to each other.

Let (u_{0X}, v_{0X}, w_{0X}) and $(u_{0X'}, v_{0X'}, w_{0X'})$ be the ideal terrestrial rectangular coordinates of the true position X and those of the assumed position X' corresponding to $(x_{X'}, y_{X'}, z_{X'})$ of (31). Further, let $\Delta u_{0X'}, \Delta v_{0X'}, \Delta w_{0X'}$ be the corrections which should be applied to the assumed position so as to obtain the true position. Then, the following equations hold:

$$\begin{aligned} u_{0X} &= u_{0X'} + \Delta u_{0X'}\,, \\ v_{0X} &= v_{0X'} + \Delta v_{0X'}\,, \\ w_{0X} &= w_{0X'} + \Delta w_{0X'}\,. \end{aligned} \tag{36}$$

On substituting u_{0X}, v_{0X}, w_{0X}, given by (36), for u_0, v_0, w_0 in (35), and rearranging the equation, we get the observation equation

$$A\cdot\Delta u_{0X'} + B\cdot\Delta v_{0X'} + C\cdot\Delta w_{0X'} = D\,, \tag{37}$$

with

$$\begin{aligned} A &= (m'_{0X}n''_{0X} - n'_{0X}m''_{0X})\,, \\ B &= (n'_{0X}l''_{0X} - l'_{0X}n''_{0X})\,, \\ C &= (l'_{0X}m''_{0X} - m'_{0X}l''_{0X})\,, \\ D &= -\{A(u_{0X'} - u_{0S}) + B(v_{0X'} - v_{0S}) + C(w_{0X'} - w_{0S})\}\,. \end{aligned}$$

As (37) has three unknowns, three sets of simultaneous observations are needed at least.

For such a practical problem as the location of islands relative to the main-land, it is usual to ask for the position expressed by the existing geodetic coordinates.

In order to rewrite (37) in terms of the geodetic coordinates, we substitute the right hand sides of the following equations, which are obtained by differentiating (7), for $\Delta u_{0X'}$, $\Delta v_{0X'}$, $\Delta w_{0X'}$ in (37).

$$\begin{aligned} \Delta u_{0X'} &= -(N+h_{X'})\sin\varphi_{X'}\cos\lambda_{X'}\cdot\Delta\varphi_{X'} - (N+h_{X'})\cos\varphi_{X'}\sin\lambda_{X'}\cdot\Delta\lambda_{X'} \\ &\quad + \cos\varphi_{X'}\cos\lambda_{X'}\cdot\Delta h_{X'}\,, \\ \Delta v_{0X'} &= -(N+h_{X'})\sin\varphi_{X'}\sin\lambda_{X'}\cdot\Delta\varphi_{X'} + (N+h_{X'})\cos\varphi_{X'}\cos\lambda_{X'}\cdot\Delta\lambda_{X'} \\ &\quad + \cos\varphi_{X'}\cdot\sin\lambda_{X'}\cdot\Delta h_{X'}\,, \\ \Delta w_{0X'} &= (N(1-e^2)+h_{X'})\cos\varphi_{X'}\cdot\Delta\varphi_{X'} + \sin\varphi_{X'}\cdot\Delta h_{X'}\,, \end{aligned} \tag{38}$$

where $\Delta\varphi_{X'}$, $\Delta\lambda_{X'}$ are given in radian, and the terms higher than the order of e^2 are omitted.

In order to denote $\Delta\varphi_{X'}$, $\Delta\lambda_{X'}$ in unit of length (meter), we introduce the radius of curvature of meridian $R_m = N^3(1-e^2)/a^2$ and the radius of parallel of latitude $R_P = N\cos\varphi$. Then, the following relations between their unit hold:

$$\begin{aligned} \Delta\varphi\ (\text{in meter}) &= \Delta\varphi\cdot R_m\ (\text{in radian})\,, \\ \Delta\lambda\ (\text{in meter}) &= \Delta\lambda\cdot R_P\ (\text{in radian})\,. \end{aligned} \tag{39}$$

With the aid of (39), (38) can be written as

$$\begin{aligned} \Delta u_{0X'} &= a\cdot\Delta\varphi_{X'} + d\cdot\Delta\lambda_{X'} + f\cdot\Delta h_{X'}\,, \\ \Delta v_{0X'} &= b\cdot\Delta\varphi_{X'} + e\cdot\Delta\lambda_{X'} + g\cdot\Delta h_{X'}\,, \\ \Delta w_{0X'} &= c\cdot\Delta\varphi_{X'} \qquad\qquad + i\cdot\Delta h_{X'}\,, \end{aligned} \tag{40}$$

with the notations of

$$\begin{aligned} a &= -(1-e^2\sin^2\varphi_{X'})\cdot\sin\varphi_{X'}\cos\lambda_{X'}/(1-e^2)\,, & e &= \cos\lambda_{X'}\,, \\ b &= -(1-e^2\sin^2\varphi_{X'})\cdot\sin\varphi_{X'}\sin\lambda_{X'}/(1-e^2)\,, & f &= \cos\varphi_{X'}\cos\lambda_{X'}\,, \\ c &= \ \ (1-e^2\sin^2\varphi_{X'})\cdot\cos\varphi_{X'}\,, & g &= \cos\varphi_{X'}\sin\lambda_{X'}\,, \\ d &= -\sin\lambda_{X'}\,, & i &= \sin\varphi_{X'}\,. \end{aligned}$$

Inserting (40) into (37), and normalizing the coefficients of $\Delta\varphi_{X'}$, $\Delta\lambda_{X'}$, $\Delta h_{X'}$, we finally get the observation equations (the equations of position planes) expressed by the geodetic coordinates

$$P \cdot \Delta\varphi_{X'} + Q \cdot \Delta\lambda_{X'} + R \cdot \Delta h_{X'} = K, \tag{41}$$

with

$$\begin{aligned} A \cdot a + B \cdot b + C \cdot c &= p, \\ A \cdot d + B \cdot e \quad &= q, \\ A \cdot f + B \cdot g + C \cdot i &= r, \\ \sqrt{p^2+q^2+r^2} &= n, \end{aligned}$$

$$P = p/n, \qquad Q = q/n, \qquad R = r/n, \qquad K = D/n.$$

In these equations, K means the distance between the position plane and the assumed position of unknown station.

7. Reduction of simultaneous method

In Fig. 1b, let (x_A, y_A, z_A) be the coordinates of station A, and (l_A, m_A, n_A), (l_X, m_X, n_X) be the direction cosines of satellite as seen from stations A and X at time t, respectively. Then, we can find their values by the same procedure as described previously. (Here it should be noted that station X is equipped with a timing device).

In this case, as is clearly seen from Fig. 1b, the position plane is expressed by the equation

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ l_A & m_A & n_A \\ l_X & m_X & n_X \end{vmatrix} = 0. \tag{42}$$

As (42) is given by the sidereal coordinate system, we must transform the coordinate system into the ideal terrestrial rectangular coordinate system by (5), from the same reason as mentioned in the trailing method.

Then, we get the following equation instead of (42).

$$\begin{vmatrix} u_0-u_{0A} & v_0-v_{0A} & w_0-w_{0A} \\ l_{0A} & m_{0A} & n_{0A} \\ l_{0X} & m_{0X} & n_{0X} \end{vmatrix} = 0. \tag{43}$$

Next, let $(\Delta u_{0X'}, \Delta v_{0X'}, \Delta w_{0X'})$ be the corrections for the coordinates of the assumed unknown station $(u_{0X'}, v_{0X'}, w_{0X'})$ in order to get the corresponding true coordinates (u_{0X}, v_{0X}, w_{0X}).

Analogous to (37), expanding (43) by $\Delta u_{0X'}$, $\Delta v_{0X'}$, and $\Delta w_{0X'}$, we obtain

$$A' \cdot \Delta u_{0X'} + B' \cdot \Delta v_{0X'} + C' \cdot \Delta w_{0X'} = D', \tag{44}$$

with the notations

$$\begin{aligned} A' &= m_{0A} \cdot n_{0X} - n_{0A} \cdot m_{0X}, \\ B' &= n_{0A} \cdot l_{0X} - l_{0A} \cdot n_{0X}, \\ C' &= l_{0A} \cdot m_{0X} - m_{0A} \cdot l_{0X}, \\ D' &= -\{A'(u_{0X'} - u_{0A}) + B'(v_{0X'} - v_{0A}) + C'(w_{0X'} - w_{0A})\}. \end{aligned}$$

To express (44) by the geodetic corrections $(\Delta\varphi_{0X'}, \Delta\lambda_{0X'}, \Delta h_{0X'})$ instead of $(\Delta u_{0X'}, \Delta v_{0X'}, \Delta w_{0X'})$, we can employ the same procedure as (38), (39), (40). Then, we get the observation equation

$$G' \cdot \Delta\varphi_{0X'} + H' \cdot \Delta\lambda_{0X'} + I' \cdot \Delta h_{0X'} = J', \tag{45}$$

with

$$
\begin{aligned}
A'\cdot a+B'\cdot b+C'\cdot c &= g' \\
A'\cdot d+B'\cdot e &= h' \\
A'\cdot f+B'\cdot g+C'\cdot i &= i' \\
\sqrt{g'^2+h'^2+i'^2} &= n'
\end{aligned}
$$

$$
G'=g'/n', \quad H'=h'/n', \quad I'=i'/n', \quad J'=D'/n'
$$

where the coefficients $a, b, \cdots, i$ are given by (40).

In the previous paper (Yamazaki, 1968), the author pointed out that the planetary aberration effect is more serious in the simultaneous method than in the trailing method.

To take account of this effect, the author employed a relative correcting method which uses the value at time $\left(t+\frac{\Delta_A}{c}-\frac{\Delta_X}{c}\right)$ as the direction of satellite as seen from A, without any corrections for station X. Here Δ_X and Δ_A mean the distances between the satellite and stations X and A, respectively, and c means the light velocity. As for the values of Δ_X and Δ_A, it will be sufficient to use those given by the prediction. On the other hand, the displacements due to the diurnal aberration are generally so small that they are negligible. If necessary, the corrections can be applied to the positions of satellite according to the procedure commonly adopted in spherical astronomy.

8. Reduction of simultaneous-trailing method

As was discussed in section 2, the position line S_0X (Fig. 1c) can be obtained by only a set of simultaneous observations in the case of simultaneous-trailing method. However, it is more convenient to solve the problems by observation equations given in the form of the position plane instead of the position line, because this procedure makes it possible to treat together with the results of the observations made by other methods. As those position planes, we take the planes AS_0X and BS_0X (Fig. 1c), whose intersection yields the position line S_0X.

In this method, A and X are equipped with a precise timing device. Then, we can apply the simultaneous method for the results of simultaneous observations made at A and X, by which the position plane AS_0X is obtained.

On the other hand, we do not know the direction of satellite as seen from B at time t, because B is not equipped with the timing device. However, we can calculate it according to the method described in section 6, for a set of observations made at A and B. This brings the same result as the case in which B is equipped with the precise timing device, too. Combining this calculated direction of satellite as seen from B with the observed one as seen from A, and applying the simultaneous method, we get the position plane BS_0X.

9. Concluding remarks

So far, we have seen that all of the observation equations for the three kinds of satellite triangulation methods can be given as equations of position planes. Accordingly, we can solve the results of the simultaneous observations made by these different methods together, provided that these observations are independent to each other. But, here, we must pay attention to the followings.

As pointed out in the previous paper (Yamazaki, 1968), the trailing method has a serious disadvantage that the resulting accuracy depends on the geometrical relation of the base stations to the satellite track, though this method has such advantages that ordinary astrographs are applicable and precise timing devices are unnecessary. The same is true in the simultaneous-trailing method, although the unknown station needs the precise timing device. Thus, if we want to solve these observation equations together, the weights should be assigned to them, as is explained in the Appendix 1.

Then, it should be noted that either of the trailing method and the simultaneous-trailing method needs the two base stations, of which the coordinates are known. Therefore, in these methods, we only get the positions relative to the two base stations belonging to the same geodetic system. For the purposes of controlling the existing geodetic system and determining the absolute position, the simultaneous method only gives a effective direct means.

As is evidently seen from Fig. 1 b, two sets of two-station observations made by the simultaneous method make possible to determine the direction between the stations (the reader is referred to Aardoom, et al., 1966). Starting from an initial point, we can successively locate all stations by the simultaneous method, while a scale factor remains as unknown. At present, transcontinental traverses by electronic distance measuring devices are used to scale the satellite triangulation, but they may be replaced by laser ranging of satellites in future.

Now, we have seen that the coordinates of the stations derived from the satellite triangulation are given by the three-dimensional coordinate system which origin may generally not coincide with the center of gravity of the earth, although its axes are parallel to those of the ideal terrestrial rectangular coordinate system. In order to bring the origin to the center of gravity of the earth, it needs to combine the results obtained by the geometrical method described in this paper with those by the dynamical methods which analyze the motion of artificial satellites in the gravitational field of the earth.

This is a dissertation presented for the degree of Doctor of Philosophy at Kyoto University.

Appendix

1. Weighting of the position plane

In the previous paper (Yamazaki, 1968), the author pointed out that it is unfavorable to assign the equal weight to the position planes derived from different methods, say, the trailing and the simultaneous methods, and proposed the formulae appropriate to these weights by means of vector analysis. In the following, these formulae will be derived employing the notations in this paper.

1) Trailing-method

In the previous paper, the weight W is given by the following formulae

$$W=1/\sqrt{(E_A^2+E_B^2+E_X^2)} \quad ,$$

with

$$\begin{aligned} E_A&=[\{(\boldsymbol{b}\cdot\boldsymbol{r}_{BA})/(\boldsymbol{b}\cdot\boldsymbol{a})\}^2+\{(\boldsymbol{a}\cdot\boldsymbol{x})(\boldsymbol{b}\cdot\boldsymbol{r}_{BA})/(\boldsymbol{b}\cdot\boldsymbol{a})^2\}^2]^{1/2}\cdot\varepsilon_A\,,\\ E&=[\{(\boldsymbol{a}\cdot\boldsymbol{x}).|\boldsymbol{r}_{BA}|/(\boldsymbol{b}\cdot\boldsymbol{a})\}^2+\{(\boldsymbol{a}\cdot\boldsymbol{x})(\boldsymbol{b}\cdot\boldsymbol{r}_{BA})/(\boldsymbol{b}\cdot\boldsymbol{a})^2\}^2]^{1/2}\cdot\varepsilon_B\,,\\ E_X&=|\boldsymbol{r}_{X'S}|\times\varepsilon_X\,. \end{aligned} \tag{i}$$

Here, the notations mean: (Fig. 1a)

- $\boldsymbol{a}$: unit vector of the direction of satellite as seen from A
- $\boldsymbol{b}$: unit vector of normal of the plane BS_1S_2
- $\boldsymbol{x}$: unit vector of normal of the plane XS_1S_2
- $\boldsymbol{r}_{BA}$: relative position vector between A and B
- $\boldsymbol{r}_{X'S}$: relative position vector between an assumed position X' of X and satellite S
- ε_A, ε_B and ε_X : observation errors of satellite at station A, B and X, respectively.

The factors in the right hand side of (i) are expressed by notations of the present paper as follows:

$$\begin{aligned} (\boldsymbol{a}\cdot\boldsymbol{x})&=l_AL_X+m_AM_X+n_AN_X\,,\\ (\boldsymbol{b}\cdot\boldsymbol{a})&=l_AL_B+m_AM_B+n_AN_B\,,\\ (\boldsymbol{b}\cdot\boldsymbol{r}_{BA})&=L_B(x_B-x_A)+M_B(y_B-y_A)+N_B(z_B-z_A)\,,\\ |\boldsymbol{r}_{BA}|&=\{(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2\}^{1/2}=\Delta_{BA}\,,\\ |\boldsymbol{r}_{X'S}|&=\{(x_S-x_{X'})^2+(y_S-y_{X'})^2+(z_S-z_{X'})^2\}^{1/2}=\Delta_{X'}\,, \end{aligned} \tag{ii}$$

where

$$L_X=A/(A^2+B^2+C^2)^{1/2}\,,\quad M_X=B/(A^2+B^2+C^2)^{1/2}\,,\quad N_X=C/(A^2+B^2+C^2)^{1/2}\,,$$
$$L_B=L/(L^2+M^2+N^2)^{1/2}\,,\quad M_B=M/(L^2+M^2+N^2)^{1/2}\,,\quad N_B=N/(L^2+M^2+N^2)^{1/2}\,.$$

2) Simultaneous-method

In the previous paper, the weight W is given by the following formulae.

$$W=\{(\varepsilon_A^2+\varepsilon_X^2)^{1/2}\cdot|\boldsymbol{r}_{AX'}|/\sin\phi\}^{-2}\,, \tag{iii}$$

where ϕ means the angle subtended between straight lines AS_0 and XS_0 (Fig. 1b), and $\boldsymbol{r}_{AX'}$ means the relative position vector between an assumed position X' and A. They are also expressed by the notations of the present paper as follows:

$$\begin{aligned} \sin\phi&=(A'^2+B'^2+C'^2)^{1/2}\,,\\ |\boldsymbol{r}_{AX'}|&=\{(x_{X'}-x_A)^2+(y_{X'}-y_A)^2+(z_{X'}-z_A)^2\}^{1/2}\,. \end{aligned} \tag{iv}$$

3) Simultaneous-trailing method

In the previous paper, the author did not derived the formulae for this case. Analogous to the other cases, we can easily derive the formulae.

Let us denote the unit vector of the direction of satellite as seen from B by $\boldsymbol{S}$, the position vector of B by $\boldsymbol{r}_B$, and its satellite range by Δ_B (Fig. 1c).

Then, the position vector of satellite $\boldsymbol{r}_S$ is given by

$$\boldsymbol{r}_S=\boldsymbol{r}_B+\Delta_B\cdot\boldsymbol{S}\,. \tag{v}$$

On the other hand, $\boldsymbol{r}_S$ is given as the intersection point of the direction of satellite as seen from A with the trail of satellite as seen from B by ((5) in the previous paper)

$$\boldsymbol{r}_S=\boldsymbol{r}+\boldsymbol{a}_A(\boldsymbol{b}\cdot\boldsymbol{r}_{BA})/(\boldsymbol{b}\cdot\boldsymbol{a})\,, \tag{vi}$$

where $\boldsymbol{r}_A'$ means the position vector of A and others have the same meanings as in (i), (ii).

Differentiating (v) and (vi), we can derive the reduced positioning error $\delta\boldsymbol{S}$ of satellits as seen from B as follows:

$$\delta\boldsymbol{S}=\left\{\frac{(\boldsymbol{b}\cdot\boldsymbol{r}_{AB})}{(\boldsymbol{b}\cdot\boldsymbol{a})}\cdot\delta\boldsymbol{a}-\frac{(\boldsymbol{b}\cdot\delta\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{r}_{AB})\boldsymbol{a}}{(\boldsymbol{b}\cdot\boldsymbol{a})^2}+\frac{\{(\delta\boldsymbol{b}\cdot\boldsymbol{r}_{AB})(\boldsymbol{b}\cdot\boldsymbol{a})-(\boldsymbol{b}\cdot\boldsymbol{r}_{AB})(\delta\boldsymbol{b}\cdot\boldsymbol{a})\}\boldsymbol{a}}{(\boldsymbol{b}\cdot\boldsymbol{a})^2}\right\}\Big/\Delta_B, \tag{vii}$$

with the assumptions $\delta r_B=0$, and $\delta r_A=0$.

We have already seen in section 8 of the present paper, that the simultaneous-trailing method can be divided into two simultaneous methods with the aid of the calculated S. In regard to the resulting weight, we can also use the same formula as (iii) for the combination of B and X, that is,

$$W=[\{(e_B)^2+(\varepsilon_X)^2\}^{1/2}\cdot|r_{X'B}|/\sin\phi]^{-2}. \quad \text{(viii)}$$

In the above, all factors excepting e_B have the same meanings as the notations in (iii), and e_B is the resulting error which can be calculated by the following formula from the observation error ε_B of the station B.

$$e_B=|\delta S|=\left[\left\{\frac{(b\cdot r_{AB})^2}{(b\cdot a)^2}+\frac{(b\cdot r_{AB})^2}{(b\cdot a)^4}\right\}\cdot\varepsilon_A^2 + \left\{\frac{|r_{AB}|^2}{(b\cdot a)^2}+\frac{(b\cdot r_{AB})^2}{(b\cdot a)^4}\right\}\cdot\varepsilon_B^2\right]^{1/2}\Big/\Delta_B. \quad \text{(ix)}$$

The relation between the notations in (viii), (ix) and those of the present paper may be analogized from the relations (ii) and (iv).

2. Corrections for the shape of satellite

In the observations of balloon satellites, the center of photographic image of satellites may not generally coincide with the true center of the satellite, due to the phase angle at the satellite between the sun and the observing station. For instance, in the case of satellite of diameter about 40 m, such as Echo II, this displacement reaches to about 20 m at the phase angle of 180° on calculation, though we may not, of course, observe for such extreme case.

In general, however, this effect is not so serious, because in the satellite triangulation we treat the relative discrepancy between the satellite's directions as seen from every station. However, for the satellite having larger diameter and lower altitude, this effect could not be ignored.

On the existing balloon satellites, we have at present very little informations about their exact shape in flight. However, if we assume that the satellites have spherical shape and property of specular reflection, the problem can easily be solved as below. (Even for diffuse reflection we can take the same solution within the accuracy concerned, if the observations are not made at extreme phase angle.)

See Fig. 2 Let S be the sun, A the center of the satellite, O the observer, and M the

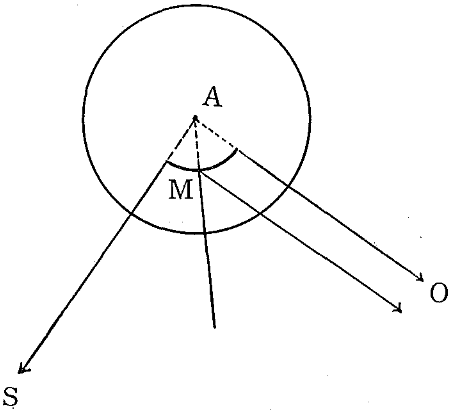

Fig. 2. Phase Effect

reflection point for the specular reflection. By the law of reflection, we have

$$\angle SAM = \angle MAO .$$

Let us denote the apparent direction of the satellite as seen from O by $\overrightarrow{OM}$. Then, the problem is reduced to determining the corresponding true direction $\overrightarrow{OA}$ by applying corrections for $\overrightarrow{OM}$.

Let $X_\odot$, $Y_\odot$, $Z_\odot$ be the geocentric equatorial rectangular coordinates of the sun (which are given in the ephemeris), x_A, y_A, z_A those of the satellite (for which the predicted values can be used with sufficient accuracy), and x_0, y_0, z_0 those of the station.

We now consider a new rectangular coordinate system by taking parallel to the above system and has the origin at the center of the satellite.

Let $l_\odot$, $m_\odot$, $n_\odot$ and l_0, m_0, n_0 be the direction cosines of the sun and station 0, respectivel, as seen from the center of the satellite in the new coordinate system. Their values can be calculated by the following equation

$$\begin{aligned} l_\odot &= (X_\odot - x_A)/D_\odot \approx X_\odot/D_\odot , \\ m_\odot &= (Y_\odot - y_A)/D_\odot \approx Y_\odot/D_\odot , \\ n_\odot &= (Z_\odot - z_A)/D_\odot \approx Z_\odot/D_\odot , \end{aligned} \tag{x}$$

$$\begin{aligned} l_0 &= (x_0 - x_A)/D_A , \\ m_0 &= (y_0 - y_A)/D_A , \\ n_0 &= (z_0 - z_A)/D_A , \end{aligned} \tag{xi}$$

with

$$D_\odot \approx \sqrt{X_\odot^2 + Y_\odot^2 + Z_\odot^2} ,$$

$$D_A = \sqrt{(x_0 - x_A)^2 + (y_0 - y_A)^2 + (z_0 - z_A)^2} .$$

Similarly, we write l_M, m_M, n_M for the direction cosins of the point M. Since the vector $\overrightarrow{SM}$ is on the plane OAS and $\angle OAM = \angle SAM$, the law of reflection givens the following equations

$$\begin{vmatrix} l_M & m_M & n_M \\ l_\odot & m_\odot & n_\odot \\ l_0 & m_0 & n_0 \end{vmatrix} = 0 , \tag{xii}$$

and

$$l_M l_\odot + m_M m_\odot + n_M n_\odot = l_M l_0 + m_M m_0 + n_M n_0 . \tag{xiii}$$

By rearranging (xii), (xiii), we obtain

$$\begin{aligned} & l_M L + m_M M + n_M N = 0 , \\ & l_M(l_\odot - l_0) + m_M(m_\odot - m_0) + n_M(n_\odot - n_0 = 0 , \end{aligned} \tag{xiv}$$

with

$$\begin{aligned} L &= (m_\odot n_0 - n_\odot m_0) , \\ M &= (n_\odot l_0 - l_\odot n_0) , \\ N &= (l_\odot m_0 - m_\odot l_0) \end{aligned} \tag{xv}$$

On the other hand, rightascension and declination of M as seen from A, α_M, δ_M, are related to l_M, m_M, n_M by

$$\begin{aligned} l_M &= \cos \alpha_M \cdot \cos \delta_M , \\ m_M &= \sin \alpha_M \cdot \cos \delta_M , \\ n_M &= \sin \delta_M . \end{aligned} \tag{xvi}$$

Inserting (xvi) into (xiv) and rearranging, we get

$$\text{Tan}\,\alpha_M = \frac{\{L(n_\odot - n_0) - N(l_\odot - l_0)\}}{\{N(m_\odot - m_0) - M(n_\odot - n_0)\}}$$
$$\text{Tan}\,\delta_M = -\frac{(\cos\alpha_M \cdot L + \sin\alpha_M \cdot M)}{N}, \tag{xvii}$$

where the sign of α_M is discriminated by the condition $|\alpha_\odot - \alpha_M| \gtrless 6^h$. Thus we can calculate the values of l_M, m_M, n_M from (xvii) and (xvi).

Now let x_M, y_M, z_M be the geocentric equatorial rectangular coordinates of . Then M their values are given by the following equations

$$\begin{aligned} x_M &= x_A + r \cdot l_M , \\ y_M &= y_A + r \cdot m_M , \\ z_M &= z_A + r \cdot n_M , \end{aligned} \tag{xviii}$$

where r means the radius of the satellite.

Also, let, l'_M, m'_M, n'_M be the observed direction cosines of the satellite (corresponding to $\overrightarrow{OM}$ in Fig. 2) and l'_A, m'_A, n'_A the true ones (corresponding to $\overrightarrow{OA}$ in Fig. 2). Then, their values can be calculated by the following equations from the coordinates of A, M and O:

$$\begin{aligned} l'_M &= (x_M - x_0)/D_M , & l'_A &= (x_A - x_0)/D_A , \\ m'_M &= (y_M - y_0)/D_M , & m'_A &= (y'_A - y_0)/D_A , \\ n'_M &= (z_M - z_0)/D_M , & n'_A &= (z_A - z_0)/D_A , \end{aligned} \tag{xix}$$

where D_M means the distance between the reflection point M and the observer 0, which can be calculated by

$$D_M = \sqrt{(x_M - x_0)^2 + (y_M - y_0)^2 + (z_M - z_0)^2} . \tag{xx}$$

Clearly, the corrections $\Delta l'_M$, $\Delta m'_M$, $\Delta n'_M$ which should be applied to l'_M, m'_M, n'_M in order to get l'_A, m'_A, n'_A are given by

$$\Delta l'_M = l'_A - l'_M , \quad \Delta m'_M = m'_A - m'_M , \quad \Delta n'_M = n'_A - n'_M . \tag{xxi}$$

We can now proceed to show these corrections by the values of right ascention and declination $\Delta\alpha'_M$, $\Delta\delta'_M$.

Let α'_M, δ'_M be the observed rightascension and declination of the satellite. They are related to l'_M, m'_M, n'_M by the following equations:

$$\tan\alpha'_M = \mathrm{m}'_M / l'_M , \qquad \sin\delta'_M = n'_M . \tag{xxii}$$

Differentiating (xxii), we get

$$\begin{aligned} \Delta\alpha'_M &= (l'_M \cdot \Delta m'_M - m'_M \cdot \Delta l'_M)\cos^2\alpha'_M / l'^2_M , \\ \Delta\delta'_M &= \sec\delta'_M \cdot \Delta n'_M . \end{aligned} \tag{xxiii}$$

Finally, we get the corrected true positions of the satellite α'_A, δ'_A from the equations

$$\begin{aligned} \alpha'_A &= \alpha'_M + \Delta\alpha'_M , \\ \alpha'_A &= \delta'_M + \Delta\delta'_M . \end{aligned} \tag{xxiv}$$

Here it is to be noted that in the trailing method the directions of the satellite's trails as seen from B and X (Fig. 1a), which are unequipped with timing device, are shown by the standard coordinates (ξ_i, η_i, $i=1$, $\cdots$, N) instead of the celestial coordinates (α_i, δ_i, $i=1$, $\cdots$, N). In this case, accordingly, the procedure for the above corrections must be taken after transformations to the celestial coordinates.

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